ABSTRACT
Different from the prediction by the Sabine formula, in non-diffuse sound fields, reverberation times also depend on the room shape, the distribution of absorption coefficients and especially on the scattering coefficients. In long rectangular rooms with absorbing side walls they depend very sensitively – almost inversely proportionally - on the scattering coefficients of the front and back wall and may be much longer than the Sabine value. For this special case, a semi-analytical approach had been found by considering the absorption and scattering fractions in a quasi-one-dimensional reverberation process in a 2D rectangle. This has now been compared with ray tracing results.

Different from the announcement in the previous abstract, this work has not been extended to 3D but to other 2D-shapes, investigating the effects of geometry especially 'wall' inclinations. Now also the well-known effect of wall inclination and scattering on the uprising of flutter echoes has been investigated quantitatively. Therefore, the Dietsch echo criterion has been utilized, an algorithm to compute the audibility of echoes from impulse responses or echograms. These are computed by sound particle simulations.

Special and critical cases are circular or cylindrical rooms with their focus effects, where the reverberation times may be much shorter than the Sabine value. A very topical and critical case is the draft of the concert hall in the new Elbphilharmonie in Hamburg.

1 INTRODUCTION
In contrast to common reverberation theory, the reverberation times RT in non-diffuse sound fields depend on the room shape, the distribution of the absorption and especially the scattering coefficients $\sigma$. However, the RT can up to now only be computed numerically or for totally diffusely reflecting walls (DSF). Due to the non-mixing of the sound field pre-dominant sound directions arise, in cases of plan-parallel walls also the feared flutter-echoes. These effects can be computed quite well by ray tracing methods. However the aim of the presented investigations (some of them done a longer time ago and just summarized here) has been to find rather practical rules, relationships between room geometry and mainly surface scattering and RT and flutter echoes. So, also a quantitative measure is needed; therefore the Dietsch echo criterion (DEC) computable from echograms has been utilized. Other room acoustical parameters also depend on room geometry (most important to mention the lateral efficiency important for music perception on the room’s ground plan) but these are not investigated here.

The investigations, by part in 2D, are intended rather as studies of relationships and sensitivities than exact computations. They are based on energetic models, for only one imagined frequency band. This paper concentrates on the results and is organized as follows:
- a semi-analytical approach to compute the RT in the most critical case in an rectangular room with totally absorbing side walls and totally reflecting front walls as a function of the scattering coefficient $\sigma$;
- an even analytical formula for the RT as a function of the scattering of the ceiling in a semi-
circular room;
- some new examples of new results of 2D sound particle simulations in different shapes of
'rooms' investigating the dependencies of the RT and DEC from different geometries especially
wall inclinations; the simulation procedure and the definition of the DEC are briefly described;
- some examples of results of 3D sound particle simulations in a cylindrical room investigating
the dependencies of the Definition (Deutlichkeit) and the DEC from the scattering coefficients of
the walls;
- finally some older but still surprising results of the RT as a function of the ceiling inclination and
reflector position in the draft for the concert hall of the Elbphilharmonie in Hamburg.

2 REVERBERATION IN THE DIFFUSE SOUND FIELD
In diffuse sound fields, according the Sabine theory the RT does only depend on the room
volume \( V \) and the equivalent absorption area \( A = \sum_{n=1}^{N} \alpha_n S_n \) (\( S_n \) = single of \( N \) surfaces, \( \alpha_n \) = absorption degrees):

\[
T_{sab} = 6 \cdot \log(10) \frac{A}{c \alpha_m} = 0.163 \frac{V}{A}
\]  
(1)

where in the derivation

\[
\Lambda = 4V/S
\]  
(2)

is the mean free path length (S= total surface) and the mean absorption coefficient is

\[
\alpha_m = \sum_{n=1}^{N} \alpha_n S_n / S = \Lambda / S
\]  
(3)

The Eyring RT, preferred here, is (with \( c= \) sound velocity)

\[
T_{Eyr} = 6 \cdot \log(10) \frac{\Lambda}{c \alpha_m} \approx 0.163 \frac{V}{S \alpha_m}
\]  
(4)

where the absorption exponent is

\[
\alpha_m' = -\ln(1 - \alpha_m)
\]  
(5)

In 2D these formulae are modified to

\[
T_{sab} = 6 \cdot \ln(10) \frac{\Lambda}{c \alpha_m} \approx 0.128 \frac{S}{B}
\]  
(6)

and

\[
T_{Eyr} = 6 \cdot \log(10) \frac{\Lambda}{c \alpha_m} \approx 0.128 \frac{S}{U \alpha_m}
\]  
(7)

where the mean free path length is \( \Lambda = \pi S/U \), \( S= \) ground area, \( U= \) circumference, the average
absorption degree is \( \alpha_m = \sum_{n=1}^{N} \alpha_n b_n / U = B/U \) with edge lengths \( b_n \), \( B= \) eff. absorption length.
It shall be reminded that the crucial pre-condition for these well-known formulae is a constant
illumination of all surfaces and hence the 'diffuse sound field' defined as the state of a totally
isotropic and homogenous sound propagation strictly given only for the case of zero absorption
and totally Lambert diffuse reflections and only approximately for 'low' average absorption
(actually everywhere low absorption!) and at least partially diffusely reflecting walls [1].

The scattering coefficient \( \sigma \) of a surface is defined as the energy scattered into other directions
than the direction of the geometric reflected energy related to the energy reflected in total ; the
scattering may be due to surface roughness (as considered here) but also due to edge
diffraction. For simulations with sound particles, scattering may be implemented either directly
drawing a random number \( z \), and if \( z < \sigma \) , the particle is scattered, or with the vector mixing
model [2] where, according \( \sigma \) the geometric directional vector and a randomly determined
directional vector according Lambert are interpolated. \( \sigma = 1 \) means maximum diffuse, i.e.
Lambert reflection with the probability density

\[
\frac{dp}{d\theta} = \cos(\theta)/2
\]  
(8)

here normalized for 2D ( \( \theta= \) reflection angle to the normal).
3 REVERBERATION IN A RECTANGULAR ROOM

To simplify the task, in the ‘room’ of length $L$ and height $H$ ‘floor’ and ‘ceiling’ are totally absorbing, front- and end walls totally reflecting and scattering with $\sigma$ (fig.1). Parameters are: the proportion $q=H/L$ (typically $< 1$ and scattering coefficient $\sigma$). The source is in the middle (fig.1). With these and the above definitions, the Sabine RT is

$$T_{\text{Sab}} = 0.064 \cdot q \cdot L = 0.064 \cdot H$$

and for better reference the (much smaller) Eyring RT is

$$T_{\text{ey}} = \frac{T_{\text{Sab}}}{(1+q) \ln(1+1/q)} = \frac{T_{\text{Sab}}}{-\ln(q)}$$

The computational model is that of a quasi—one-dimensional reverberation: if the rectangle is relatively long ($q<0$) and its ‘front’ (short) ‘walls’ are reflecting, the other totally absorbing, then the sound is travelling mainly in longitudinal ($x$) direction, where the reflections (‘flutter echoes’) happen almost in constant time intervals of $L/c$ and $L$ is much longer than the mean free path length $\Lambda$ in a diffuse sound field. Absorption takes place only indirectly, by scattering from the front walls to the ‘side walls’ (‘floor’ and ‘ceiling’). To handle the only partially diffuse reflections, the sound energies are classified into:

- the geometrical reflected ‘ur’-radiation from the source (‘s’);
- the always diffusely reflected energy (‘d’) and
- the, after once diffusely, $j$ times geometrically reflected energies (‘g’)

Now, there are set up certain ‘transition coefficients’ between these classes and between energies of $k$th and $(k+1)$th reflection order [3].

The idea to estimate the energies of the ur-radiation $E_s(k)$ is: after the time $t$, or the distance $ct$ only those energies are not absorbed at the ‘side walls’ which propagated into the direction of the angle fraction $4\phi/(2\pi)$ determined by $H/2$ in a distance of $(k-1/2)$ mirror room lengths (Fig.1). Then this energy decays with :

$$E_s(t) = \frac{2}{\pi} \arctan \left( \frac{H}{2ct} \right) \text{ or } E_s(k) = \frac{2}{\pi} \arctan \left( \frac{q}{2k-1} \right) \sim \frac{1}{k}$$

and is not exponential!

Fig.1: the rectangular room (in the middle) and its mirror rooms only in longitudinal direction as only front walls reflecting

The fraction of scattered energy reaching a ‘side’ wall from a front wall (Fig.2) – another transition coefficient- is according the ‘radiosity’ method determined by the integral

$$g_{HL} = \frac{1}{H} \int_0^H \int_0^\pi L \cos^2 \beta \cos \theta \, d\theta \, dx = \frac{1}{2q} \left( 1 + q - \sqrt{1 + q^2} \right)$$

The transition coefficient from a diffuse reflection to absorption after some geometric reflections at the front walls can be computed in an analogous way by instead aiming at the absorbing sidewall of a mirror room (Fig.1)

Fig.2: Diffuse energy interchange between a front and a side wall (dashed circle indicates the Lambert reflection) ; $x$ and $y$ are counted from the upper left corner,
Correspondingly, the energies in the different classes summarized win or lose energy in every step. The RT of the total process is computed from the decay of the summed-up energies.

Just some results room length for L=10m, for proportions q=H/L=0.5 and for σ = 0.1 (typical low scattering) shall be discussed here. Fig. 3 shows the result The Eyring (and Kuttruff RTs) are very low (0.19s, 0.16s). The RT for the geometrically reflected ur-energy (without scattering) is very much higher: for a -30dB -threshold and without scattering T60=9.36s ! As the levels do not decay linearly, this "RT" depend on the threshold up to which the decay is counted. In this example, for a 20dB -threshold and with 10% scattering, the result is T60=1.4s. The slopes of the decays and hence the RT for the urgeo, diffuse and total energies are very similar as the diffuse energies are ‘fed’ and hence dominated by the ur-geometric radiation. After a very few reflections, the energies decay ‘in parallel’ - just on different levels depending on σ (compare the red, blue and black line in fig.4). It further turned out that the classes of once diffuse and then multiple geometric reflections (violet lines in Fig. 3) only weakly influence the result of the total RT as these reflections are on a very low level in all cases, only the very first orders need to be considered.

The RT in such a rectangular room drastically increase with decreasing scattering coefficients of the front walls (see fig. 4). For all q<1, the RT can be estimated by

\[ T \approx T_{\text{diff}} \sigma^{-0.8} \text{ with } T_{\text{Diff}} \approx 6 \cdot \ln(10) \frac{L}{c \cdot \ln(2)} = \frac{0.635}{q \cdot (\ln(2) - \ln(q))} \cdot T_{\text{Sab}} > T_{\text{Sab}} \text{ for } q=1 \]  

So, in shoe-box-rooms with front walls of low absorption and scattering, quasi-one-dimensional reverberation (‘flutter echoes’) lets the RT be very much longer than according to Sabine almost inverse –proportionally to their scattering coefficient σ. Even for σ = 1 is T>TSab.

<table>
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<th>σ</th>
<th>0.05</th>
<th>0.1</th>
<th>0.25</th>
<th>0.5</th>
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<td>1.8s</td>
<td>0.85s</td>
<td>0.49s</td>
<td>0.28s</td>
</tr>
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</table>
4 REVERBERATION IN A SEMI-CIRCULAR ROOM

The opposite is the case with focusing effects onto absorbing parts of the surface (often the audience) as in domes or semi-circular rooms [3]. To have a chance for an analytical formula $RT(\sigma)$ the problem is simplified as much as possible (see Fig.5): there are only two geometric parameters: the radius $r$ and the width $2b<<r$ of a small piece of floor around the center (parameter $q=b/r$) which is totally absorbing; all other ‘surfaces’ are totally reflecting, with the scattering degree $\sigma$. The source is close over the ground in the center. If a ‘sound particle’ hits the ground, it is either absorbed if $|x|<b$ else reflected but re-emitted from the center. The approach is here to ask: Which effective absorption coefficient $\alpha_{eff}$ does a ‘sound particle’ ‘see’ on the floor if it is reflected from the ceiling according the scattering coefficient $\sigma$? This is inserted into the Sabine RT Eqn. 6. For reference, following from geometry, the Sabine RT is

$$T_{sab} = 0.100 \cdot r^2/b$$

(Fig.5 Model of the semi-circular room with radius $r$: The absorbing part of the floor $2b$ is seen from a reflection point $R$ over an angle of $2\beta$; the yellow bubble at $R$ indicates the mixed Lambert reflection directivity for a scattering coefficient of $\sigma$ (angle range $\pm \sigma \cdot \pi/2$)

To estimate the hit probability onto the absorbing part of the floor and the effective absorption coefficient, an approximate probability distribution for the semi-diffuse reflection $\frac{dp}{d\vartheta}$ is derived from the Lambert law for $\frac{dp}{d\vartheta}$ (Equ. 8) and the scattering coefficient $\sigma$ by applying the mentioned vector mixing model. The angle of the mixed reflection can be approximated very well by a linear relationship $\gamma \approx \sigma \cdot \vartheta$. The probability density for the mixed reflection law is:

$$p' = \frac{dp}{d\vartheta} = \cos(\gamma/\sigma)/(2\sigma)$$

(15)

The absorbing part $2b$ is seen over an angle of twice

$$\beta \approx b \cdot \cos \vartheta/r$$

(16)

and, on average over all $\vartheta$,

$$\beta_m \approx 2/\pi \cdot q$$

(17)

The probability that the absorbing part is hit is the integral over the reflection distribution

$$P_\alpha(q,\sigma) \approx \sin(\min(2/\pi \cdot q, \pi/2)/\sigma) = \alpha_{eff}$$

(18)

As absorption happens in this ‘room’ only every two reflections, $\Lambda \approx 2\beta$ and the $\alpha_{eff}$ are inserted into Eqn.6 instead of $\Lambda$ and $\alpha_m$. This yields: $T_{semi} \approx 0.128 \sigma \cdot \frac{r^2}{b} = 1.28 \cdot \sigma \cdot T_{sab}$

(19)

This is a RT formula as a direct and linear function of the scattering coefficient $\sigma$ of the ceiling!

As many approximations have been used in this derivation, a sound particle simulation was started for comparison (3000 particles). The reverberation time was computed from a linear regression analysis of the level decay of the whole room energy in the range 0…-30dB. As an example, figure 6 shows the graphs of the RTs as a function of the scattering coefficients compared with ray tracing results: Even for a wide absorbing part of the floor $q=b/r=0.5$ (with $r=10m$, $T_{sab}=2s$) the agreement between sound particle simulations with the result of Eqn. 19 is quite good – except for low scattering values of $\sigma < 0.1$.

The RT approaches zero with totally geometric reflections of the ceiling – caused by the focusing effect onto the absorbing audience in the middle- and approaches about the Sabine value for totally diffuse reflections – both not surprising.
5. DEPENDENCIES OF REVERBERATION TIMES AND ECHOES FROM ROOM GEOMETRY, ABSORPTION AND SCATTERING

To investigate further relationships, there is no chance for analytical formulae. Therefore some sound particle simulations have been performed.

5.1. The sound particle simulation method (SPSM)

The tracing algorithm of the sound particle method is the same as with the well known ray tracing. As well geometrical as diffuse reflections may be handled with these straight-forward energetic methods. Only the method to evaluate the local sound intensities or rather energy densities is different. As point shaped particles never hit receiver points exactly some spatially extended detectors around the receiver points have to be established. The energies of crossing particles are weighted with the inner crossing distances. Thus, due to the spatial extension of the detectors, there is a certain temporal uncertainty in the computed echograms, but there is no directivity of the detectors, their sensitivity is only dependent on their volume. So, also cubic detectors are allowed. A 2D-grid of those cubes forms an ‘audience area’ (fig.7). This is also computationally most effective [2].

By evaluating the respective well known energy proportions, room acoustical parameters such as definition or clarity, centre time and echo criterion, by evaluating also incident directions, lateral efficiencies, and by regression analysis of the level decay, local RT can be evaluated and displayed as a coloured map (see below).

5.2. The Dietsch Echo Criterion (DEC)

As the degree of definition only depends on time integrals, it is not sensitive to a temporal coincidence of reflected energy as with echoes. Echoes are the more audible and disturbing the later they arrive. To evaluate this quantitatively, Dietsch [4] proposed to evaluate a modified build-up-function of the center time $\tau_s$:

$$\tau_s(\tau) = \int_0^\tau I_m(t) \cdot t \cdot dt / \int_0^\tau I_m(t) \cdot dt$$

then $EC(\tau) = \Delta \tau_s(\tau) / \Delta \tau_e$ and $DEC = \max(EC(\tau))$ (20)

In the integral over the intensity I, each echo causes a jump $\Delta \tau_s$ of $\tau_s$, so the maximum is evaluated to describe something like an overall disturbance by echoes; this is the Dietsch echo criterion DEC. The integral in the denominator accounts for the fact that echoes are the less audible the more reflections were preceding. For speech, $m=1/3$ , $\Delta \tau_e = 9ms$ , for music $m=1/2$ and $\Delta \tau_e = 14ms$ . An echo is evaluated as disturbing by 10% of listeners, if $EC>0.9$, by more than 50%, if $EC>1$(for music >1.5 or >1.8 respectively).
5.3. Some results of a cylindrical room (in 3D)
Cylindrical rooms are a worse case in room acoustics due to their focusing effects. Of course, an aim is then to break up this effect by scattering wall surfaces. The question is, to what extent the typical lack of Definition and the severe flutter echoes in the middle of such rooms can be diminished. An occasion for thorough investigations and improvements had been the building of the new parliament room in Bonn (former capital of West-Germany) [5].

Fig 8: Focusing of rays in a circle room opposite the source

To study the principle effects, some high accuracy sound particle simulations (100000 SP) had been made specialized for an idealized cylindrical room of 40m diameter and 10m height, with source position 4m from the center, absorption 50% at ceiling and floor (Sabine reverberation time 1.5 s at 1kHz). By the way, without scattering, it is important whether an exact cylinder is simulated or any polygonal room. Here only some typical results are shown (figures 9):

Figures 9: Sound fields in a 40m diameter cylindrical room: left: distribution of Definition $D$, middle: echograms $EC(\tau)$ near the focus; right: distribution of the Dietrich Echo Criterion ($DEC$); side walls specular reflecting (upper figures), with 50% diffusivity degree (lower figures). The colors are ordered as in a rainbow: towards red: high values, towards violet: low values.

For $D$, from green on the values are over 50%, hence good; for $DEC$, from green on more than 50% of listeners feel disturbed. In the middle, the dashed white line indicates the $EC=1$ value, over which more than 50% of the listeners feel disturbed.

The Definition distribution with specular reflecting walls is extremely unfavorable: Just where mostly the audience is situated, opposite to the source, an extended deficit region is found, with a 'black hole' near the focus (upper left figure upper part, compare fig.8). With 50% diffusely reflecting side walls, the $D$ distribution looks different (sickle shaped) but the average $D$ value is hardly better (54% instead of 50%). This can be explained analytically by the Fermat principle: Specular reflections are those of shortest detour. All diffuse reflections have longer detours. Thus, the average degree of Definition can never be enhanced by such means. As it was
computed numerically, the relative extent of such a deficit area for D in rooms of the same ground area is largest in circular rooms; it vanishes in quadratic or even 1:2 rectangular rooms.

Without measures at the walls, the many echoes near the focus are disastrous, as the echogram or rather the EC-function (upper middle of figure 9) indicates. With 50% scattering walls the echogram is much smoothed ("the echoes dissolve in the time scale"), but still one first echo remains audible. The DEC-distribution in the unaltered room is quite 'cleft'; on most places (94%, except the blue/violet fields) strong echoes occur, the DEC is over 50% - awful. With 50% scattering walls the situation is much improved, only 10% bad places are left.

Summarized, in circular rooms by making the walls scattering the average value of definition can hardly be improved - a large deficit area remains never reached by reflections less than 50ms delayed. This is an unavoidable deficit of large circular rooms (with reflecting walls) – even with optimum reverberation times. Echoes can be considerably diminished – but hardly totally avoided.

**Practical consequences:** Cylindrical or spherical rooms should be decomposed into as many scattering secondary structures as possible. Investigations showed the following preference:
- Absorbers are the only measure influencing the average D, but their influence on the EC is poor; often not practicable, also diminishing the desired lateral efficiency;
- diffusers are preferable; they do not enhance D but diminish the EC considerably;
- better are reflectors with optimized orientation; best turned by a vertical and inclined by a horizontal axis; to avoid additional focus points, no constant swing angles!

However, even with drastic alterations of secondary structures in circular rooms, disturbing echo effects cannot be destroyed completely, so, **circular auditoria best should be avoided at all!**

### 5.4. Some results of the crucial shape of the Elbphilharmonie concert hall in Hamburg

According the modern trend, the concert hall of the ‘Elbphilharmonie’ is (as in Berlin) a vineyard hall (orchestra in the middle, 2150 seats around). The crucial point is: it is tent shaped, maximum height 30m, volume $V \approx 25000 \, m^3$, so $V/N \approx 12 \, m^3/\text{seat}$ let expect the RT to be after $T_{\text{Sab}} = 0.163 \frac{V}{A} \approx 0.23 \frac{V}{N} \approx 2.7 \, s$. Before he warned the Hamburg authorities, this motivated the author to perform some 2D sound particle simulations on the dependencies of the RT from the shape of the roof.

Only some principle effects shall be mentioned here. First the effect of the ceiling inclination with a flat absorbing ground (audience, $a_{\text{aud}} = 0.58$, else $a_0 = 0.05$, rectangle: $12m^2 \cdot 40m$, $T_{\text{EY}} = 2s$, $T_{\text{Sab}} = 2.5s$). The RTs decrease with the roof angle by a factor of 1.5 in the (theoretical) case of specular reflecting walls, but even for a $\sigma = 5 \ldots 10^{10}$ as realistic for large smooth walls, the decreasing factor is still 1.5 (green curve). The effect

![Diagram showing RTs for different roof angles](image-url)
Fig. 11 Relative prolongation factors in relation to the Eyring reverberation time as a function of the roof inclination angle (left), with the scattering degrees $\sigma$ (DG) as a parameter. The same in the following.

This reason for this effect is: in the rectangular 'room' with plan-parallel specular reflecting end walls, a long surviving longitudinal reverberation occurs (see section 3), the level decay is not linear but hanging' (see fig. 3). With increasing scattering this effect vanishes, i.e. the Eyring RT value is reached. Interesting is: The same is achieved with a roof inclination – but only almost and only if $45^\circ$. The corresponding echograms (computed for a typical receiver on the ground) in fig. 12 show clearly: both scattering or roof inclination destroys the flutter echoes and makes the sound field diffuser and thus makes the RT approach the Eyring value.

Another, more surprising effect is the strong dependency of the RT from the roof inclination in the case of a steep ($45^\circ$) audience (as planned).

The RT is maximum (1.5 times higher than the Eyring value) for a $7.5^\circ$ roof angle (for $\sigma = 0.1$ blue line). The reason for that can be identified as caused by a focussing of sound onto the reflecting stage in the middle (bottom) only in the case of the $7.5^\circ$ -shape while in the two other cases rather the (absorbing) audience left and right is illuminated. As an especially amazing (in this project highly relevant) fact, it shall just be reported, that with the 'peak' in the $6^\circ$ inclined roof (12m wide and 4m high, figure in the middle) the RT drastically drops again by a factor of 2 -under the Eyring value. Such a peak seems to act as a magic diffusor! But with the planned large stage reflector hanging from the top (fig. 10) the RT again increases by a factor 1.5! [6]
6 CONCLUSION
Reverberation times and echoes drastically depend on the scattering coefficients: in the rectangular room they decrease while they increase in the semi-circular room (almost inversely respectively directly proportionally to the scattering coefficients). Such RT formulae remain only a rough estimation and a study for special cases. Otherwise numerical simulations are necessary. In a cylindrical rooms, making the walls scattering, helps to destroy the worse flutter-echoes, but there will principally remain a lack of definition in the middle.
A general tendency is: After a very few reflections, in all reasonable cases (of non-zero-scattering) the energy decays are dominated by diffuse reflections. So, to predict scattering coefficients remains a very important and challenging task.

6 OUTLOOK
Many other parameters could be altered. Some of the investigations should be extended to 3D. A possible example of high relevance for noise immission prognosis could be the application to 'street canyons' with parallel and partially diffusely reflecting house facades which can be considered as a quasi-2D-problem (with a totally absorbing 'heaven' and a mirroring street). It remains as a challenge to reliably compute the scattering coefficients. Possibly, some results of further investigations will be presented orally at the conference.

7 REFERENCES